



High density effective theory results for two and three massive flavors color-superconductivity

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We describe some topics related to massive two and three flavors color superconductivity (CSC), obtained in the framework of high density effective theory (HDET); moreover we present a *modified* Nambu-Jona Lasinio model with running - as opposite to fixed - ultraviolet cutoff, discussing the troubles that one encounters with the former when discussing QCD at high density.

1 Introduction

The existence of color-superconductivity (CSC) at very large densities and very low temperatures is a well established consequence of QCD[1]. The study of this state of quark and gluons matter is not only interesting for itself, but could be very important to achieve a greater knowledge of the physics of compact stars. Indeed, the conditions of high density-low temperature could be realized for example in the core of a neutron star.

While at asymptotic densities one can use perturbative methods, at lower densities (as for example the ones that could be realized in the core of a neutron star) one can use effective field theories as Nambu-Jona Lasinio (NJL) models. Moreover, the phase diagram of the possible phases of CSC with massless flavors is well known, while the situation with massive quarks is under investigation.

In this paper we address to study different aspects of CSC with massive flavors, employing a modified NJL model and working in the framework of HDET[7] (for a review see[10]). As a first point, we consider the three flavor case, known in literature as color-flavor locked (CFL) phase, with massless u, d quarks and massive s quark. We are concerned with:

- the role of the ultraviolet cutoff in the NJL models
- the relevance of the effects due to the strange quark mass

Moreover, we briefly face to the following problems for the two massive flavors (2SC) phase:

- study of the gap equation

- dynamical properties of the pseudo-Goldstone η' .

2 A modified NJL model

This section and the following one are based on our paper[12].

When the NJL interaction is used for modelling QCD at vanishing temperature and density, one can fix the UV cutoff Λ such as to get realistic quark constituent masses; in this way one also gets the value of the NJL coupling constant G for the choosen scale of energy Λ [15]. In such a case Λ is thought of as fixed once for all. This gives no problems at zero density; however, leads to difficulties when one tries to simulate QCD at finite chemical potential. In fact, in HDET one takes as relevant degrees of freedom all the fermions with momenta in a shell around the Fermi surface. The thickness of the shell is measured by a cutoff δ , which is the cutoff for momenta measured from the Fermi surface. The cutoff δ satisfies the bounding condition $\Delta \ll \delta \ll \mu$, and is related to the NJL cutoff by means of the relation $\Lambda = \delta + \mu$.

This relation is problematic when one is interested in the behavior of the theory for varying μ . The constraint $\Lambda = \mu + \delta$ would force δ to vanish for increasing μ , starting from $\mu < \Lambda$, which does not correspond to the asymptotic behaviour ($\mu \rightarrow \infty$) QC behaviour[14]. The uncorrect behavior of Δ arises because the model is taken to be valid only for momenta up to Λ which forbids to go to values of μ of the order or higher than Λ . Clearly this constitutes an obstacle in physical situations where the typical chemical potential is about 400 or 500 MeV (e.g. in compact stellar objects) with a δ of the order 150 or 200 MeV. In fact it turns out difficult, if not impossible, to explore higher values of μ for any reasonable choice of Λ .

We make a proposal to overcome this situation¹, suggesting the following procedure that allows the introduction of a *running* NJL coupling constant. We write the Nambu-Jona Lasinio equations with a three dimensional cutoff Λ ². From the f_π equation (with $f_\pi = 93$ MeV) we get the function $m^* = m^*(\Lambda)$ which we use in the m^* equation to get the function $G = G(\Lambda)$. The result of this analysis is in fig. 1. Our choice implies that a NJL model is defined at any

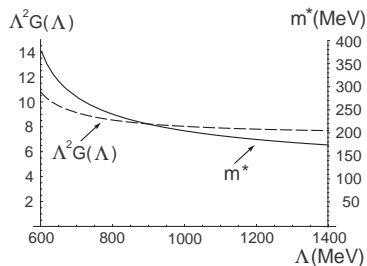


Figure 1. The running NJL coupling constant G (dashed line) and the running constituent mass m^* (in MeV, solid line) as functions of Λ (in MeV). Λ is the ultraviolet cutoff. The vertical axis on the left refers to $\Lambda^2 G(\Lambda)$, while the axis on the right refers to m^* .

scale by using the appropriate $G(\Lambda)$. Whereas in the usual case we have to keep the momenta smaller than the cutoff, now, for any given momenta, we can fix the cutoff in such a way that it is much bigger than the momenta.

In applying these considerations to the calculations at finite density, we have only to use the appropriate value of the coupling as given by $G(\mu + \delta)$. To give an explicit example we consider the CFL phase with massless quarks. There are two independent gaps Δ and Δ_9 ($\Delta_9 = -2\Delta$ if the pairing is only in the antitriplet channel) and the gap equations are [2, 10]:

$$\begin{aligned} \Delta &= -\frac{\mu^2 G}{6\pi^2} \left(\Delta_9 \operatorname{arcsinh} \frac{\delta}{|\Delta_9|} - 2\Delta \operatorname{arcsinh} \frac{\delta}{|\Delta|} \right), \\ \Delta_9 &= -\frac{4\mu^2 G \Delta}{3\pi^2} \operatorname{arcsinh} \frac{\delta}{|\Delta|}. \end{aligned} \quad (1)$$

If one uses a fixed value for $\Lambda = \mu + \delta$, as for instance in ref. [16], one gets a non monotonic behavior of the gap, as it can be seen from fig. 2, (dashed line); a similar behavior was found in [16] (their fig. 1), albeit a different choice of the parameters produces some numerical differences. On the other hand the solid line

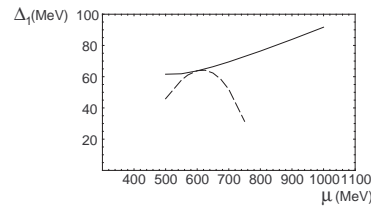


Figure 2. The CFL gap Δ for massless quarks, as obtained from eq. 1, versus the quark chemical potential for the two cases discussed in the text. Solid line: running NJL coupling $G(\mu + \delta)$ and cutoff $\delta = c\mu$, with $c = 0.35$; dashed line: $\delta = \Lambda - \mu$, where $\Lambda = 800$ MeV, and $G(\Lambda) = 13.3$ GeV⁻². The picture shows the different qualitative behavior with μ of the gap.

shows an increasing behavior of the gap. We see that in this way one reproduces qualitatively the behavior found in QCD for asymptotic chemical potential [14]. This result is obtained by the running NJL coupling $G(\mu + \delta)$, with the following choice of the cutoff δ :

$$\delta = c\mu \quad (2)$$

with c a fixed constant ($c = 0.35$ in fig. 2). The reason for this choice is that, as discussed above, when increasing μ , we do not want to reduce the ratio of the number of the relevant degrees of freedom to the volume of the Fermi sphere. Requiring the fractional importance to be constant is equivalent to require eq. (2).

3 The CFL phase with massive strange quark

The massive case is considerably more involved, and the simple set of equations (1) has to be substituted by a system of 5 equations [3] that can be only solved numerically. In [16] the CFL phase with massive strange quark was also considered. In comparison with [3] the derivation we present here has the advantage of offering semi-analytical results, thanks to an expansion in powers of m_s/μ . We principally differ from ref. [16] for the different treatment of the cutoff, as discussed in the previous section, and for the inclusion of pairing in both the antitriplet and the sextet color channel. The possibility of a semi-analytical treatment rests on the HDET approximation. This effective lagrangian approach was extended in [11] to the 2SC phase with massive quarks and here we treat the three flavor case.

Because of limited space we discuss here only the fundamental results; formal details may be found in [12].

We consider the HDET formulation of the theory of CFL condensation in both the antisymmetric $\bar{\mathbf{3}}_A$ and in the symmetric $\mathbf{6}_S$ channels. We assume equal

¹For the sake of discussion we consider here the ideal case of massless quarks, the more realistic case of a massive strange quark will be considered in the following.

²For a detailed account see Ref. [15] and Eqs.(1) and (2) of [12].

masses (actually zero) for the up and down quarks and neglect quark-antiquark chiral condensates, whose contribution is expected to be negligible in the very large μ limit³. The condensate we consider is therefore

$$\begin{aligned} \langle 0 | \psi_{\alpha i} C \gamma_5 \psi_{\beta j} | 0 \rangle &\sim (\Delta_{ij} \epsilon^{\alpha\beta I} \epsilon_{ij I} \\ &+ G_{ij} (\delta^{\alpha i} \delta^{\beta j} + \delta^{\alpha j} \delta^{\beta i})) \end{aligned} \quad (3)$$

The first term on the r.h.s accounts for the condensation in the $\mathbf{\bar{3}}_A$ channel and the second one describes condensation in $\mathbf{6}_S$ channel. As we assume $m_u = m_d$, we put

$$\begin{aligned} \Delta_{us} = \Delta_{ds} &\equiv \Delta & \Delta_{ud} &\equiv \Delta_{12} \\ G_{uu} = G_{du} = G_{ud} = G_{dd} &\equiv G_1 & G_{us} = G_{ds} &\equiv G_2 \\ G_{ss} &\equiv G_3 \end{aligned} \quad (4)$$

which reduces the number of independent gap parameters to five. We stress that we impose electrical and color neutrality, which, as shown in ref. [4], is indeed satisfied in the color-flavor locked phase of QCD because in this phase the three light quarks number densities are equal $n_u = n_d = n_s$, with no need for electrons, i.e. $n_e = 0$. As a consequence, the Fermi momenta of the three quarks are equal:

$$p_{F,u} = p_{F,d} = p_{F,s} \equiv p_F, \quad (5)$$

Armed with the fermionic lagrangian, one writes the five gap equations whose numerical solutions are reported in Fig.3 for $m_s = 250$ MeV and $c = 0.35$ [12].

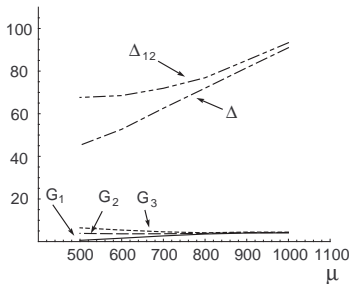


Figure 3. The five gap parameters vs. the chemical potential μ ; results obtained by the numerical solution of the exact gap equations, for $m_s = 250$ MeV and $\delta = 0.35\mu$. The upper curve refers to Δ_{12} , while the middle one refers to Δ . The lower curves are the data obtained for the three sextet gap parameters $|G_1|$, $|G_2|$ and $|G_3|$. Gaps and μ are expressed in MeV.

³Also the contribution from the repulsive $\mathbf{6}_S$ channel is expected to be small, but we include it because the gap equations are consistent only with condensation in both the $\mathbf{6}_S$ and the $\mathbf{\bar{3}}_A$ channels.

One notes that the effects of the strange quark mass are of order m_s^2/μ^2 , so a semi-analytical solution can be found by performing an expansion in the strange quark mass $x_s = \frac{m_s}{\mu_s} \approx \frac{m_s}{\mu} \ll 1$. Here $\mu = \mu_u = \mu_d$, $\mu_s = \mu + \delta\mu$ with $\delta\mu = \mathcal{O}(m_s/\mu)^2$. As shown in Ref. [12], in this perturbative frame the set of five gap equations reduces to a set of five *algebraic* equations plus the two transcendental gap equations for the massless case. In Ref. [12] we find that the agreement between the exact and the approximated solutions of the gap equations is indeed good.

4 The 2SC phase with massive flavors

Briefly we describe some results obtained within the formalism of HDET for the 2SC phase of QCD with u and d massive quarks. Such a phase is characterized by the condensation pattern

$$\langle 0 | \psi_{\alpha i} C \gamma_5 \psi_{\beta j} | 0 \rangle \sim \Delta \epsilon^{\alpha\beta 3} \epsilon_{ij 3} \quad (6)$$

We work in the hypothesis $p_{F,u} = \mu_u v_u = p_{F,d} = \mu_d v_d$, which implies a difference in the chemical potentials of the two flavors when their masses are unequal. In this case one has only one gap parameter, and the solution of the gap equation may be written explicitly [11].

We wish to discuss here some property of the $U(1)_A$ pseudo-Goldstone mode of the 2SC phase. It is well known that the condensation pattern in Eq.(6) breaks spontaneously $U(1)_A$, which is however also broken by the strong anomaly. At high μ the latter breaking is soft and one expects that the associated pseudo-Goldstone, called η' , is almost massless. In HDET one gets the dynamical properties of the pseudo-Goldstone by bosonization of fermionic lines [11]. This results in a one-loop effective lagrangian. As noticed in [11], [8] and [20] the mass term of the η' is due to antiquark insertions, while the gap insertions give rise to the kinetic term. Within the HDET one obtains the mass formula (see also [8], [20]):

$$m_{\eta'}^2 = 4 \Delta^2 \frac{m_u m_d}{\mu^2} \log \left(\frac{\mu}{\Delta} \right) \quad (7)$$

To this result one should add the contribution of the instantons. However, the latter has been already estimated [17], [19], [18] and the result is that at asymptotic μ the instanton contribution is negligible with respect to the quark massive one (other interesting studies of the role of instantons in high density or large N_c QCD may be found in [21]).

Moreover, we get formulas for the η' velocity and its decay constant

$$v_{\eta'}^2 = \frac{|v_u||v_d|}{3} \quad f_{\eta'}^2 = \frac{8\mu_u\mu_d}{\pi^2} \frac{|v_u||v_d|}{|v_u| + |v_d|} \quad (8)$$

In particular, the $v_{\eta'}^2$ should be compared with the massless value $v_{\eta'}^2 = 1/3$ [10]. The last equations are new results, which go beyond the expansion in m/Δ commonly used in the computation of the mass effects in CSC.

5 Conclusions and outlooks

In this paper we proposed a modified NJL model with running coupling constant, which allows to overcome the troubles of the zero density fixed cutoff scheme. Moreover, we have discussed several points relative to massive quarks effects in two CSC phases of QCD, in the framework of HDET.

For the CFL phase with a massive strange quark we have numerically solved the gap equations for condensation both in the triplet and in the sextet channels. Such a solutions explicitly show that the massive effects are of order $x_s^2 = m_s^2/\mu^2$; as a consequence, a perturbative expansion in terms of x_s^2 is meaningful. The perturbative gap equations are a set of algebraic equations, as opposite to the full gap equations which are transcendental. We show in Ref.[12] that there exists a good agreement between the exact and the perturbative gap equations. This perturbative approach could be applied for example to get analytic results on the massive states of CFL, as for example gluons (in a recent paper[22] a non perturbative study of the massive spectrum of the CFL phase is presented).

Turning to the 2SC phase, we have solved the gap equation for both massive flavors: in HDET one gets a closed formula which relates the gap parameter to the quark masses.

Lastly but not least important, we shown HDET results for the dynamical properties of the pseudo-Goldstone η' , obtaining a leading order mass formula and expressions for velocity and decay constant which go beyond the usual m/Δ expansion used in the massive effects in the CSC calculations.

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